

PSEUDORANDOMNESS OF THE OSTROWSKI SUM-OF-DIGITS FUNCTION

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ABSTRACT. For an irrational $\alpha \in (0, 1)$, we investigate the Ostrowski sum-of-digits function σ_α . For α having bounded partial quotients and $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$, we prove that the function $g : n \mapsto e(\vartheta \sigma_\alpha(n))$, where $e(x) = e^{2\pi i x}$, is pseudorandom in the following sense: for all $r \in \mathbb{N}$ the limit

$$\gamma_r = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)}$$

exists and we have

$$\lim_{R \rightarrow \infty} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r|^2 = 0.$$

1. INTRODUCTION AND MAIN RESULTS

Let g be an arithmetical function. The set of $\beta \in [0, 1)$ satisfying

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n < N} g(n) e(-n\beta) \right| > 0$$

is called the *Fourier–Bohr spectrum* of g .

The function g is called *pseudorandom in the sense of Bertrandias* [4] or simply *pseudorandom* if the limit

$$\gamma_r = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)}$$

exists for all $r \geq 0$ and the family γ is zero in quadratic mean, that is,

$$\lim_{R \rightarrow \infty} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r|^2 = 0.$$

(We note that by the Cauchy–Schwarz inequality this is equivalent to $\frac{1}{R} \sum_{r < R} |\gamma_r| = o(1)$ for bounded g .) Pseudorandomness can be understood as a property of the *spectral measure* associated to g : Assume that the autocorrelation γ of g exists. By the Bochner representation theorem there exists a unique measure μ on the Torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ such that

$$\gamma_r = \int_{\mathbb{T}} e(rx) d\mu(x)$$

for all r . Then g is pseudorandom if and only if the discrete component of μ vanishes. We refer to [10] for more details.

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It is known that pseudorandomness of a bounded arithmetic function g implies that the spectrum of g is empty, which can be proved using van der Corput's inequality. For the convenience of the reader, we give a proof of this fact in Section 2.

The converse of this statement does not always hold. However, it is true for q -multiplicative functions $g : \mathbb{N} \rightarrow \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, which has been proved by Coquet [6, 7, 8]. Here a function $g : \mathbb{N} \rightarrow \mathbb{C}$ is called q -multiplicative if $f(q^k n + b) = f(q^k n)f(b)$ for all integers $k, n > 0$ and $0 \leq b < q^k$.

The purpose of this paper is to prove an analogous statement for the Ostrowski numeration system, that is, for α -multiplicative functions. Assume that $\alpha \in (0, 1)$ is irrational. The Ostrowski numeration system has as its scale of numeration the sequence of denominators of the convergents of the regular continued fraction expansion of α . More precisely, let $\alpha = [0; a_1, a_2, \dots]$ be the continued fraction expansion of α and $p_i/q_i = [0; a_1, \dots, a_i]$ the i -th convergent to α , where $i \geq 0$. By the greedy algorithm, every nonnegative integer n has a representation

$$(1.1) \quad n = \sum_{k \geq 0} \varepsilon_k q_k$$

such that

$$\sum_{0 \leq k < K} \varepsilon_k q_k < q_K$$

for all $K \geq 0$. This algorithm yields the unique expansion of the form (1.1) having the properties that $0 \leq \varepsilon_0 < a_1$, $0 \leq \varepsilon_k \leq a_{k+1}$ and $\varepsilon_k = a_{k+1} \Rightarrow \varepsilon_{k-1} = 0$ for $k \geq 1$, the *Ostrowski expansion of n* .

For a nonnegative integer n let $(\varepsilon_k(n))_{k \geq 0}$ be its Ostrowski expansion. An arithmetic function g is α -additive resp. α -multiplicative if

$$f(n) = \sum_{k \geq 0} f(\varepsilon_k(n) q_k) \quad \text{resp.} \quad f(n) = \prod_{k \geq 0} f(\varepsilon_k(n) q_k)$$

for all n . Examples of α -additive functions are the functions $n \mapsto \beta n$ and the α -sum of digits of n [9]:

$$\sigma_\alpha(n) = \sum_{i \geq 0} \varepsilon_i(n).$$

We refer the reader to [3] for a survey on the Ostrowski numeration system. In particular, we want to note that the Ostrowski numeration system is a useful tool for studying the discrepancy modulo 1 of $n\alpha$ -sequences, see for example the references contained in the aforementioned paper.

Moreover, see [1] for a dynamical viewpoint of the Ostrowski numeration system, and [12, 2] for more general numeration systems.

Our main theorem establishes a connection between the Fourier–Bohr spectrum and pseudorandomness for α -multiplicative functions.

Theorem 1.1. *Assume that g is a bounded α -multiplicative function. The Fourier–Bohr spectrum of g is empty if and only if g is pseudorandom.*

Using a theorem by Coquet, Rhin and Toffin [11, Theorem 2], we obtain the following corollary.

Corollary 1.2. *Assume that $\alpha \in (0, 1)$ is irrational and has bounded partial quotients and $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$. Then $n \mapsto e(\vartheta \sigma_\alpha(n))$ is pseudorandom.*

In particular, this holds for the *Zeckendorf sum-of-digits function*, which corresponds to the case $\alpha = (\sqrt{5} - 1)/2 = [0; 1, 1, \dots]$. This special case can be found in the author's thesis [14].

We first present a series of auxiliary results, and proceed to the proof of Theorem 1.1 in section 3.

2. LEMMAS

We begin with the well-known inequality of van der Corput.

Lemma 2.1 (Van der Corput's inequality). *Let I be a finite interval in \mathbb{Z} and let $a_n \in \mathbb{C}$ for $n \in I$. Then*

$$\left| \sum_{n \in I} a_n \right|^2 \leq \frac{|I| - 1 + R}{R} \sum_{0 \leq |r| < R} \left(1 - \frac{|r|}{R} \right) \sum_{\substack{n \in I \\ n+r \in I}} a_{n+r} \overline{a_n}$$

for all integers $R \geq 1$.

In the definition of pseudorandomness for bounded arithmetic functions g , we do not actually need the square.

Lemma 2.2. *Let g be a bounded arithmetic function such that the correlation of g exists. The function g is pseudorandom if and only if*

$$\lim_{R \rightarrow \infty} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| = 0.$$

For the proof of sufficiency we note that we may without loss of generality assume that $|g| \leq 1$. The other direction is an application of the Cauchy-Schwarz inequality.

As we noted before, pseudorandomness of g implies that the spectrum of g is empty.

Lemma 2.3. *Let g be a bounded arithmetic function. If g is pseudorandom, then the Fourier-Bohr spectrum of g is empty.*

Proof. The proof is an application of van der Corput's inequality (Lemma 2.1). We have for all $R \in \{1, \dots, N\}$

$$\begin{aligned} \left| \frac{1}{N} \sum_{0 \leq n < N} g(n) e(n\beta) \right|^2 &\leq \frac{N - 1 + R}{RN^2} \sum_{0 \leq |r| < R} \left(1 - \frac{|r|}{R} \right) e(r\beta) \sum_{0 \leq n, n+r < N} g(n+r) \overline{g(n)} \\ &\ll \frac{1}{R} \sum_{0 \leq r < R} \left| \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} \right| + O\left(\frac{R}{N}\right). \end{aligned}$$

Let $\varepsilon \in (0, 1)$. By hypothesis and Lemma 2.2 we may choose R so large that

$$\frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| < \varepsilon^2.$$

Moreover, we choose N_0 in such a way that $R/N_0 < \varepsilon^2$ and

$$\left| \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} - \gamma_r \right| < \varepsilon^2$$

for all $r < R$ and $N \geq N_0$. Then for $N \geq N_0$ we have

$$\left| \frac{1}{N} \sum_{0 \leq n < N} g(n) e(n\beta) \right|^2 \ll \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| + \frac{1}{R} \sum_{0 \leq r < R} \left| \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} - \gamma_r \right| + O\left(\frac{R}{N_0}\right) < 3\varepsilon^2.$$

□

The following lemma is a generalization of Dini's Theorem.

Lemma 2.4. Assume that $(f_i)_{i \geq 0}$ is a sequence of nonnegative continuous functions on $[0, 1]$ converging pointwise to the zero function. Assume that $|f_{i+1}(x)| \leq \max\{|f_i(x)|, |f_{i-1}(x)|\}$. Then the convergence is uniform in x .

Proof. For $\varepsilon > 0$, for nonnegative N and $x \in [0, 1]$ we set

$$A_N(x) = \{\xi \in [0, 1] : f_N(\xi) < \varepsilon \text{ and } f_{N+1}(\xi) < \varepsilon\}.$$

Note that this is an open set. By induction, using the property $|f_{i+1}(x)| \leq \max\{|f_i(x)|, |f_{i-1}(x)|\}$, we obtain

$$A_N(x) = \{\xi \in [0, 1] : f_n(\xi) < \varepsilon \text{ for all } n \geq N\}.$$

Trivially, we have $A_N(x) \subseteq A_{N+1}(x)$. For each $x \in [0, 1]$ there is an $N(x)$ such that $f_n(x) < \varepsilon$ for all $n \geq N(x)$. Then $x \in A_{N(x)}(x)$, therefore $(A_{N(x)}(x))_{x \in [0, 1]}$ is an open cover of the compact set $[0, 1]$. Choose x_1, \dots, x_k and N_1, \dots, N_k such that $A_{N_1}(x_1) \cup \dots \cup A_{N_k}(x_k) = [0, 1]$ and set $N = \max\{N_1, \dots, N_k\}$. By monotonicity of the sets $A_N(x)$, we obtain $A_N(x_1) \cup \dots \cup A_N(x_k) = [0, 1]$, in other words, $f_n(\xi) < \varepsilon$ for all $\xi \in [0, 1]$ and all $n \geq N$. \square

Lemma 2.5. Let $(w_i)_i$ be the increasing enumeration of the integers n such that $\varepsilon_0(n) = \dots = \varepsilon_{\lambda-1}(n) = 0$. The intervals $[w_i, w_{i+1})$ constitute a partition of the set \mathbb{N} into intervals of length q_λ and $q_{\lambda-1}$, where $w_{i+1} - w_i = q_{\lambda-1}$ if and only if $\varepsilon_\lambda(w_i) = a_{\lambda+1}$.

Proof. Assume first that $\varepsilon_\lambda(w_i) = a_{\lambda+1}$. We want to show that $w_{i+1} = w_i + q_{\lambda-1}$. Let $w_i \leq n < w_i + q_{\lambda-1}$. Then the Ostrowski expansion of n is obtained by superposition of the expansions of w_i and of $n - w_i$. In particular, for $w_i < n < w_i + q_{\lambda-1}$ we have $\varepsilon_j(n) \neq 0$ for some $j < \lambda - 1$. Moreover, in the addition $w_i + q_{\lambda-1}$ a carry occurs, producing $\varepsilon_j(w_i + q_{\lambda-1}) = 0$ for $j \leq \lambda$, therefore $w_{i+1} = w_i + q_{\lambda-1}$. The case $\varepsilon_\lambda(w_i) < a_{\lambda+1}$ is similar, in which case $w_{i+1} = w_i + q_\lambda$. \square

For an α -multiplicative function g and an integer $\lambda \geq 0$ we define a function g_λ by truncating the digital expansion: we define $\psi_\lambda(n) = \sum_{i < \lambda} \varepsilon_i(n) q_i$ and

$$g_\lambda(n) = g(\psi_\lambda(n)).$$

We will need the following carry propagation lemma for the Ostrowski numeration system.

Lemma 2.6. Let $\lambda \geq 1$ be an integer and $N, r \geq 0$. Assume that $\alpha \in (0, 1)$ is irrational and let g be an α -multiplicative function. Then

$$(2.1) \quad \left| \{n < N : g(n+r) \overline{g(n)} \neq g_\lambda(n+r) \overline{g_\lambda(n)}\} \right| \leq N \frac{r}{q_{\lambda-1}}.$$

Proof. The statement we want to prove is trivial for $r \geq q_{\lambda-1}$, we assume therefore that $r < q_{\lambda-1}$. Let w be the family from Lemma 2.5. For $w_i \leq n < w_{i+1} - r$, we have $\varepsilon_j(n+r) = \varepsilon_j(n)$ for $j \geq \lambda$. It follows that

$$\left| \{n \in \{w_i, \dots, w_{i+1} - 1\} : g(n+r) \overline{g(n)} \neq g_\lambda(n+r) \overline{g_\lambda(n)}\} \right| \leq r.$$

By concatenating blocks, the statement follows therefore for the case that $N = w_i$ for some i . It remains to treat the case that $w_i < N < w_{i+1}$ for some i . To this end, we denote by $L(N)$ resp. $R(N)$ the left hand side resp. the right hand side of (2.1). For $w_i \leq N \leq w_{i+1}$ we have

$$L(N) = \begin{cases} L(w_i), & N \leq w_{i+1} - r; \\ L(w_i) + N - (w_{i+1} - r), & N \geq w_{i+1} - r. \end{cases}$$

Note that $L(N)$ is a polygonal line that lies below $R(N)$ for $N \in \{w_i, w_{i+1} - r, w_{i+1}\}$ and therefore for all $N \in [w_i, w_{i+1}]$. By concatenating blocks, the full statement follows. \square

We define Fourier coefficients for g :

$$G_\lambda(h) = \frac{1}{q_\lambda} \sum_{0 \leq u < q_\lambda} g(u) e(huq_\lambda^{-1}).$$

Lemma 2.7. *Assume that i be such that $w_{i+1} - w_i = q_\lambda$ and let $r \geq 0$. We have*

$$(2.2) \quad \sum_{h < q_\lambda} |G_\lambda(h)|^2 e(hrq_\lambda^{-1}) = \frac{1}{q_\lambda} \sum_{w_i \leq u < w_{i+1}} g_\lambda(v+r) \overline{g_\lambda(v)} + O\left(\frac{r}{q_\lambda}\right).$$

Proof.

$$\begin{aligned} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 (hrq_\lambda^{-1}) &= \frac{1}{q_\lambda} \sum_{0 \leq u, v < q_\lambda} g_\lambda(u) \overline{g_\lambda(v)} \frac{1}{q_\lambda} \sum_{0 \leq h < q_\lambda} e\left(\frac{h}{q_\lambda}(v+r-u)\right) \\ &= \frac{1}{q_\lambda} \sum_{0 \leq u, v < q_\lambda} \llbracket v+r \equiv u \pmod{q_\lambda} \rrbracket g_\lambda(u) \overline{g_\lambda(v)} \\ &= \frac{1}{q_\lambda} \sum_{w_i \leq u, v < w_{i+1}} \llbracket v+r \equiv u \pmod{q_\lambda} \rrbracket g_\lambda(u) \overline{g_\lambda(v)} \\ &= \frac{1}{q_\lambda} \sum_{w_i \leq u < w_{i+1}-r} g_\lambda(v+r) \overline{g_\lambda(v)} + O\left(\frac{r}{q_\lambda}\right) \\ &= \frac{1}{q_\lambda} \sum_{w_i \leq u < w_{i+1}} g_\lambda(v+r) \overline{g_\lambda(v)} + O\left(\frac{r}{q_\lambda}\right). \end{aligned}$$

□

Lemma 2.8. *Let $H \geq 1$ be an integer and R a real number. For all real numbers t we have*

$$\sum_{h < H} \left| \frac{1}{R} \sum_{r < R} e(r(t+h/H)) \right|^2 \leq \frac{H+R-1}{R}.$$

This lemma is an immediate consequence of the analytic form of the large sieve, see [13, Theorem 3]. This form of the theorem, featuring the optimal constant $N-1+\delta^{-1}$, is due to Selberg.

Lemma 2.9 (Selberg). *Let $N \geq 1, R \geq 1, M$ be integers, $\alpha_1, \dots, \alpha_R \in \mathbb{R}$ and $a_{M+1}, \dots, a_{M+N} \in \mathbb{C}$. Assume that $\|\alpha_r - \alpha_s\| \geq \delta$ for $r \neq s$. Then*

$$\sum_{r=1}^R \left| \sum_{n=M+1}^{M+N} a_n e(n\alpha_r) \right|^2 \leq (N-1+\delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2.$$

As an important first step in the proof of Theorem 1.1, we show that for the functions in question we have the following uniformity property.

Proposition 2.10. *Let g be a bounded α -multiplicative function. Assume that the Fourier–Bohr spectrum of g is empty, that is,*

$$\left| \sum_{0 \leq n < N} g(n) e(-n\beta) \right| = o(N)$$

as $N \rightarrow \infty$ for all $\beta \in \mathbb{R}$. Then

$$\sup_{\beta \in \mathbb{R}} \left| \sum_{0 \leq n < N} g(n) e(-n\beta) \right| = o(N).$$

Proof of Proposition 2.10. Without loss of generality we may assume that $|g| \leq 1$, since the full statement follows by scaling. We first prove the special case

$$\lim_{i \rightarrow \infty} \sup_{\beta \in \mathbb{R}} \frac{1}{q_i} \left| \sum_{0 \leq n < q_i} g(n) e(-n\beta) \right| = 0.$$

We set $h(n) = g(n) e(-n\beta)$ and

$$S_i(\beta) = \frac{1}{q_i} \sum_{0 \leq n < q_i} h(n).$$

For all $i \geq 1$ we have

$$\begin{aligned} S_{i+1} &= \frac{1}{q_{i+1}} \sum_{0 \leq b < a_{i+1}} \sum_{0 \leq u < q_i} h(u + bq_i) + \frac{1}{q_{i+1}} \sum_{0 \leq u < q_{i-1}} h(u + a_{i+1}q_i) \\ &= \frac{q_i}{q_{i+1}} \left(\sum_{0 \leq b < a_{i+1}} h(bq_i) \right) \cdot S_i + \frac{q_{i-1}}{q_{i+1}} h(a_{i+1}q_i) S_{i-1}. \end{aligned}$$

Using the recurrence for q_i , it follows that $|S_{i+1}| \leq \max\{|S_i|, |S_{i-1}|\}$. By Lemma 2.4 we obtain the statement.

We pass to the general case. We consider partial sums of $g(n) e(n\beta)$ up to N . Assume that $w_i \leq N < w_{i+1}$. We have

$$\left| \sum_{0 \leq n < N} g(n) e(n\beta) \right|^2 \leq \left| \sum_{0 \leq n < w_i} g(n) e(n\beta) \right|^2 + q_\lambda^2 + 2Nq_\lambda.$$

We apply the inequality of van der Corput (Lemma 2.1) to obtain

$$\left| \sum_{0 \leq n < w_i} g(n) e(n\beta) \right|^2 \leq \frac{N + R - 1}{R} \sum_{|r| < R} \left(1 - \frac{|r|}{R} \right) e(r\beta) \sum_{0 \leq n, n+r < w_i} g(n+r) \overline{g(n)}.$$

We adjust the summation range by omitting the condition $0 \leq n + r < w_i$. This introduces an error term $O(NR)$. Moreover, α -additive functions f satisfy Lemma 2.6, therefore we may replace g by g_λ for the price of another error term, $O(N^2 R q_{\lambda-1}^{-1})$. Using (2.2) we get

$$\begin{aligned} &\left| \sum_{0 \leq n < w_i} g(n) e(n\beta) \right|^2 \\ &\ll \frac{N}{R} \sum_{|r| < R} \left(1 - \frac{|r|}{R} \right) e(r\beta) \left(\sum_{0 \leq n < w_i} g_\lambda(n+r) \overline{g_\lambda(n)} + O(R + NR q_{\lambda-1}^{-1}) \right) \\ &\ll NR + N^2 \frac{R}{q_{\lambda-1}} + \frac{N}{R} w_i \sum_{h < q_\lambda} |G_\lambda(h)|^2 \sum_{|r| < R} \left(1 - \frac{|r|}{R} \right) e\left(r\left(\beta + \frac{h}{q_\lambda}\right)\right). \end{aligned}$$

Note that the sum over r is a nonnegative real number. This follows from the identity

$$\sum_{|r| < R} (R - |r|) e(rx) = \left| \sum_{0 \leq r < R} e(rx) \right|^2,$$

which can be proved by an elementary combinatorial argument. We use this equation and collect the error terms to get

$$(2.3) \quad \left| \frac{1}{N} \sum_{0 \leq n < N} g(n) e(n\beta) \right|^2 \ll \frac{q_\lambda^2}{N^2} + \frac{q_\lambda}{N} + \frac{R}{N} + \frac{R}{q_{\lambda-1}} + \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \left| \frac{1}{R} \sum_{0 \leq r < R} e\left(r\left(\beta + \frac{h}{q_\lambda}\right)\right) \right|^2.$$

Next, using Lemma 2.8 we get

$$(2.4) \quad \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \left| \frac{1}{R} \sum_{0 \leq r < R} e\left(r\left(\beta + \frac{h}{q_\lambda}\right)\right) \right|^2 \leq \sup_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \frac{q_\lambda + R - 1}{R}.$$

Using the special case proved before and choosing R and λ appropriately, we obtain the statement. \square

In order to establish the existence of the correlation γ_t of g , we use the following theorem [5, Théorème 4]. (Note that we defined $\psi_\lambda(n) = \sum_{0 \leq i < \lambda} \varepsilon_i(n) q_i$.)

Lemma 2.11 (Coquet–Rhin–Toffin). *Let $\lambda \geq 1$ and $a < q_\lambda$. The set $\mathcal{E}(\lambda, a) = \{n \in \mathbb{N} : \psi_\lambda(n) = a\}$ possesses an asymptotic density given by*

$$\begin{aligned} \delta &= (q_\lambda + q_{\lambda-1} [0; a_{\lambda+1}, \dots])^{-1} && \text{if } a \geq q_{\lambda-1}; \\ \delta' &= \delta(1 + [0; a_{\lambda+1}, \dots]) && \text{if } a < q_{\lambda-1}. \end{aligned}$$

Lemma 2.12. *Let g be a bounded α -multiplicative function. Then for every $r \geq 0$ the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} g(n+r) \overline{g(n)}$$

exists.

We note that the existence of the correlation was established in [5] for the special case that $g(n) = e(y\sigma_\alpha(n))$, where $e(x) = e^{2\pi i x}$.

Proof. Let $\lambda, N \geq 0$ and $r \geq 1$ and set $k = \max\{j : w_j \leq N\}$. Moreover, let $a = a(N)$ be the number of indices $j < k$ such that $w_{j+1} - w_j = q_\lambda$ and $b = b(N)$ be the number of indices $j < k$ such that $w_{j+1} - w_j = q_{\lambda-1}$. By Lemma 2.11 $a(N)/N$ and $b(N)/N$ converge, say to A and B respectively. Let λ be so large that $r/q_{\lambda-1} < \varepsilon$. Moreover, choose N_0 so large that $|A - a(N)/N| < \varepsilon q_\lambda^{-1}$, $|B - b(N)/N| < \varepsilon q_{\lambda-1}^{-1}$ and $q_\lambda/N < \varepsilon$ for all $N \geq N_0$.

Then by Lemma 2.6 we get

$$\begin{aligned} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} &= \sum_{0 \leq n < N} g_\lambda(n+r) \overline{g_\lambda(n)} + O(Nr q_{\lambda-1}^{-1}) \\ &= \sum_{0 \leq n < w_k} g_\lambda(n+r) \overline{g_\lambda(n)} + O(q_\lambda + Nr q_{\lambda-1}^{-1}), \end{aligned}$$

therefore

$$\begin{aligned} &\left| \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} - A \sum_{0 \leq n < q_\lambda} g_\lambda(n+r) \overline{g_\lambda(n)} - B \sum_{0 \leq n < q_{\lambda-1}} g_\lambda(n+r) \overline{g_\lambda(n)} \right| \\ &\ll \left| \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} - \frac{a}{N} \sum_{0 \leq n < q_\lambda} g_\lambda(n+r) \overline{g_\lambda(n)} - \frac{b}{N} \sum_{0 \leq n < q_{\lambda-1}} g_\lambda(n+r) \overline{g_\lambda(n)} \right| + 2\varepsilon \\ &= \left| \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} - \frac{1}{N} \sum_{0 \leq n < w_k} g_\lambda(n+r) \overline{g_\lambda(n)} \right| + 2\varepsilon \end{aligned}$$

$$\ll \frac{q_\lambda}{N} + \frac{r}{q_{\lambda-1}} + 2\varepsilon.$$

By the triangle inequality it follows that the values $\frac{1}{N} \sum_{n < N} g(n+r) \overline{g(n)}$ form a Cauchy sequence and therefore a convergent sequence, which proves the existence of the correlation of g . \square

3. PROOF OF THE THEOREM

Now we are prepared to prove Theorem 1.1. If g is pseudorandom, then by Lemma 2.3 its spectrum is empty. We are therefore concerned with the converse. Let $\ell \geq 0$. We denote by a the number of $i < \ell$ such that $w_{i+1} - w_i = q_\lambda$ and by b the number of $i < \ell$ such that $w_{i+1} - w_i = q_{\lambda-1}$.

Choose ε_r such that $|\varepsilon_r| = 1$ and

$$\varepsilon_r \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 e(hr q_\lambda^{-1})$$

is a nonnegative real number. Similarly choose ε'_r for $\lambda - 1$. We have

$$\begin{aligned} & \frac{1}{R} \sum_{0 \leq r < R} \left| \frac{1}{w_\ell} \sum_{0 \leq n < w_\ell} g_\lambda(n+r) \overline{g_\lambda(n)} \right| \\ &= \left| \frac{aq_\lambda}{w_\ell} \frac{1}{R} \sum_{0 \leq r < R} \varepsilon_r \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 e\left(\frac{hr}{q_\lambda}\right) \right. \\ & \quad \left. + \frac{bq_{\lambda-1}}{w_\ell} \frac{1}{R} \sum_{0 \leq r < R} \varepsilon'_r \sum_{0 \leq h < q_{\lambda-1}} |G_{\lambda-1}(h)|^2 e\left(\frac{hr}{q_{\lambda-1}}\right) \right| + O\left(\frac{ar}{w_\ell} + \frac{br}{w_\ell}\right) \\ &= \frac{1}{R} \left| \frac{aq_\lambda}{w_\ell} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \sum_{0 \leq r < R} \varepsilon_r e\left(\frac{hr}{q_\lambda}\right) \right. \\ & \quad \left. + \frac{bq_{\lambda-1}}{w_\ell} \sum_{0 \leq h < q_{\lambda-1}} |G_{\lambda-1}(h)|^2 \sum_{0 \leq r < R} \varepsilon'_r e\left(\frac{hr}{q_{\lambda-1}}\right) \right| + O\left(\frac{r}{q_{\lambda-1}}\right) \\ &\leq \frac{1}{R} \left| \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \sum_{0 \leq r < R} \varepsilon_r e\left(\frac{hr}{q_\lambda}\right) \right| \\ & \quad + \frac{1}{R} \left| \sum_{0 \leq h < q_{\lambda-1}} |G_{\lambda-1}(h)|^2 \sum_{0 \leq r < R} \varepsilon'_r e\left(\frac{hr}{q_{\lambda-1}}\right) \right| + O\left(\frac{r}{q_{\lambda-1}}\right). \end{aligned}$$

By Cauchy-Schwarz we obtain

$$\begin{aligned} & \frac{1}{R^2} \left| \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \sum_{0 \leq r < R} \varepsilon_r e\left(\frac{hr}{q_\lambda}\right) \right|^2 \\ &\leq \frac{1}{R^2} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 \sum_{0 \leq h < q_\lambda} \left| \sum_{0 \leq r < R} \varepsilon_r e\left(\frac{hr}{q_\lambda}\right) \right|^2 \\ &\leq \frac{1}{R^2} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 \sum_{0 \leq h < q_\lambda} \sum_{0 \leq r_1, r_2 < R} \varepsilon_{r_1} \overline{\varepsilon_{r_2}} e\left(h \frac{r_1 - r_2}{q_\lambda}\right) \\ &= \frac{q_\lambda}{R^2} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 \sum_{0 \leq r_1, r_2 < R} \varepsilon_{r_1} \overline{\varepsilon_{r_2}} \delta_{r_1, r_2} \end{aligned}$$

$$= \frac{q_\lambda}{R} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4,$$

similarly for $\lambda - 1$. Using Lemma 2.6, we get

$$\begin{aligned} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| &= \lim_{\ell \rightarrow \infty} \frac{1}{R} \sum_{0 \leq r < R} \left| \frac{1}{w_\ell} \sum_{0 \leq n < w_\ell} g(n+r) \overline{g(n)} \right| \\ &= \lim_{k \rightarrow \infty} \frac{1}{R} \sum_{0 \leq r < R} \left| \frac{1}{w_\ell} \sum_{0 \leq n < w_\ell} g_\lambda(n+r) \overline{g_\lambda(n)} \right| + O\left(\frac{R}{q_{\lambda-1}}\right) \\ &\leq \left[\left(\sum_{0 \leq h < q_{\lambda-1}} |G_{\lambda-1}(h)|^4 \right)^{1/2} + \left(\sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 \right)^{1/2} \right] \left(\frac{q_\lambda}{R} \right)^{1/2} + O\left(\frac{R}{q_{\lambda-1}}\right). \end{aligned}$$

Using the hypothesis of the theorem and Proposition 2.10, we get $\sup_h |G_\lambda(h)| = o(1)$ as $\lambda \rightarrow \infty$. By Parseval's identity this implies

$$\sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 = o(1).$$

By a straightforward argument we conclude that

$$\frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| = o(1)$$

as $R \rightarrow \infty$. Since g is bounded, an application of Lemma 2.2 completes the proof of Theorem 1.1.

REFERENCES

- [1] G. Barat and P. Liardet. Dynamical systems originated in the Ostrowski alpha-expansion. *Ann. Univ. Sci. Budapest. Sect. Comput.*, 24:133–184, 2004.
- [2] Guy Barat, Valérie Berthé, Pierre Liardet, and Jörg Thuswaldner. Dynamical directions in numeration. *Ann. Inst. Fourier (Grenoble)*, 56(7):1987–2092, 2006. Numération, pavages, substitutions.
- [3] Valérie Berthé. Autour du système de numération d'Ostrowski. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):209–239, 2001. Journées Montoises d'Informatique Théorique (Marne-la-Vallée, 2000).
- [4] Jean-Paul Bertrandias. Suites pseudo-aléatoires et critères d'équirépartition modulo un. *Compositio Math.*, 16:23–28 (1964), 1964.
- [5] J. Coquet, G. Rhin, and Ph. Toffin. Représentations des entiers naturels et indépendance statistique. II. *Ann. Inst. Fourier (Grenoble)*, 31(1):ix, 1–15, 1981.
- [6] Jean Coquet. Sur les fonctions q -multiplicatives pseudo-aléatoires. *C. R. Acad. Sci. Paris Sér. A-B*, 282(4):Ai, A175–A178, 1976.
- [7] Jean Coquet. *Contribution à l'étude harmonique des suites arithmétiques*. Thèse d'Etat, Orsay, 1978.
- [8] Jean Coquet. Répartition modulo 1 des suites q -additives. *Comment. Math. Prace Mat.*, 21(1):23–42, 1980.
- [9] Jean Coquet. Répartition de la somme des chiffres associée à une fraction continue. *Bull. Soc. Roy. Sci. Liège*, 51(3-4):161–165, 1982.
- [10] Jean Coquet, Teturo Kamae, and Michel Mendès France. Sur la mesure spectrale de certaines suites arithmétiques. *Bull. Soc. Math. France*, 105(4):369–384, 1977.
- [11] Jean Coquet, Georges Rhin, and Philippe Toffin. Fourier-Bohr spectrum of sequences related to continued fractions. *J. Number Theory*, 17(3):327–336, 1983.
- [12] Peter J. Grabner, Pierre Liardet, and Robert F. Tichy. Odometers and systems of numeration. *Acta Arith.*, 70(2):103–123, 1995.
- [13] Hugh L. Montgomery. The analytic principle of the large sieve. *Bull. Amer. Math. Soc.*, 84(4):547–567, 1978.
- [14] Lukas Spiegelhofer. *Correlations for Numeration Systems*. Thesis, Vienna, Austria, 2014.